INTRO TO BRANCHED COVERS
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1) Definitions and Examples

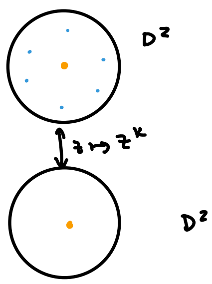
Dfn: A map $f: X^n \to Y^n$ is a branched covering if $\exists\, B^{n-2} \subseteq Y$ such that $p: X - p^{-1}(B) \to Y - B$ is a covering map.

f 1-1 on the branched locus.

B is called the branched set and $f^{-1}(B)$ is the branched locus.

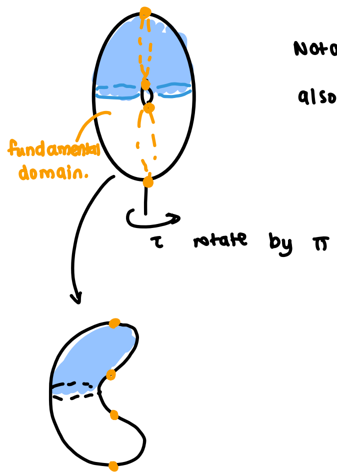
Example

① K-sheeted Branched Cover



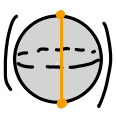
branched locus.

② quotient by group action τ

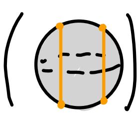


Notation: $T^2 = \Sigma_2(S^2, 4\text{ pts})$

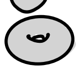
also, $S^1 \times I = \Sigma_2(D^2, 2\text{pts})$

③ $\Sigma_2\Big($  $\Big) = \Sigma_2(\bigcirc \times I) = \bigcirc \times I = B^3$

(unique)

④ $\Sigma_2\Big($  $\Big) = \Sigma_2(\bigcirc \times I)$

$= \bigcirc \times I$

$=$ 

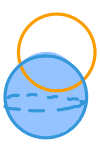
⑤ $\Sigma_2(S^3, \bigcirc)$ unknot

"the" double branched cover

means $\pi_1(Y - B) \to \mathbb{Z}/2$

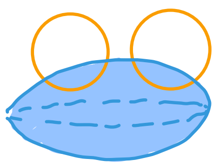
$\mu_B \to 1$

S^3:



$= \Sigma_2(\bigcirc) \cup \Sigma_2(\bigcirc)$ complement ball $= B^3 \cup B^3 = S^3.$

⑥ $\Sigma_2(S^3, \bigcirc\bigcirc)$



$\Sigma_2(S^3, \bigcirc\bigcirc) = \Sigma_2(\bigcirc) \cup_{id} \Sigma_2(\bigcirc)$ complement ball

$= D^2 \times S^1 \cup_{id} D^2 \times S^1$

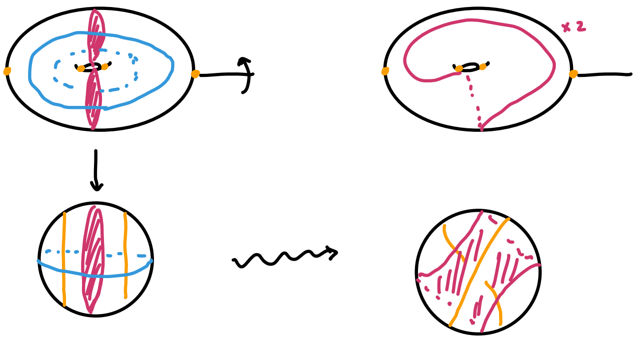
$= S^2 \times S^1$

⑦ $\Sigma_2\Big(S^3,$  $\Big) = \Sigma_2(\bigcirc) \cup \Sigma_2(\bigcirc)$

$= D^2 \times S^1 \cup_? D^2 \times S^1 = L(3,1)$

Cor:  $\not\cong 0$.

Proof of gluing map:



Theorem (Hodgson - Rubinstein) Let L be a link in S^3. Then $\Sigma_2(L) = L(p,q)$ if and only if $L = B(p,q)$ (2-bridge links)

Exercises:

1) If $K = K_1 \# K_2 \subseteq S^3$, prove $\Sigma_2(K) = \Sigma_2(K_1) \# \Sigma_2(K_2)$

2) Prove  is not a 3-fold cover of 

3) Compute $\Sigma_2($ B^4 ,  $)$

4) Compute Σ_2 ()

2) Properties / Facts

1. **Thm (Smith Conjecture)** if $\Sigma_p(S^3, K) = S^3$ and the covering is regular, then $K = 0$.

(False in dim ≥ 4)

2. **(Hilden - Lozoro - Montesinos)** Every 3-mfld (closed, connected, orientable) is a branched cover of  $\subseteq S^3$

3) Applications (to knots)

Use branched covers to study unknotting of knots.

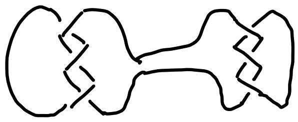
Dfn: $J \subseteq S^3$ knot. $S^3_{p/q}(J) = S^3 \setminus \nu(J) \cup D^2 \times S^1$

$$p\mu_J + q\lambda_J \longleftarrow \underset{m}{\partial D^2}$$

Note: $H_1(S^3_{p/q}(J)) = \mathbb{Z}/p$.

Thm: (Montesinos Trick) If $K \subseteq S^3$ has unknotting number 1, then $\Sigma_2(K) = S^3_{p/2}(J)$ for some $J \subseteq S^3$.

Exercise



$T_{2,3} \# -T_{2,3}$

$$\Sigma_2(K) = \Sigma_2(T_{2,3}) \# \Sigma_2(-T_{2,3})$$
$$= L(3,1) \# -L(3,1).$$

$H_1 = \mathbb{Z}/3 \oplus \mathbb{Z}/3$ → not cyclic

$\Rightarrow u(K) \geq 2.$

Proof: $\Sigma_2(K) = \Sigma_2(O) \setminus \Sigma_2($  $) \cup \Sigma_2($  $)$

$= S^3 \setminus (D^2 \times S^1) \cup D^2 \times S^1$

which is a Dehn surgery.

Exercise: $|q| = 2$. (do two half twists)

Thm (Scharlemann, Zhang)

If $K = K_1 \# K_2$, then $u(K) \geq 2$, $K_1, K_2 \neq O$

Input: Gordon - Luecke: if $S^3_{p/q}(J) = M_1 \# M_2$, $M_i \neq S^3$, then $|q| = 1$.

proof: suppose $K = K_1 \# K_2$ has $u = 1$. $\Sigma_2(K) = \Sigma_2(K_1) \# \Sigma_2(K_2)$, $\Sigma_2(K_i) \neq S^3$

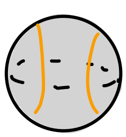
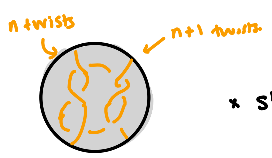
$= S^3_{\frac{p}{2}}(J)$ for some J.

But input gives $\lightning$.

Thm (Finashin - Kreck - Viro)

$\exists$ knotted surfaces $\{F_n\}_{n=1}^{\infty} \subseteq S^4$ w/ F_n's topologically isotopic, but not smoothly isotopic (exotic surfaces).

Idea of proof: Start w/ simple surface $F_1 \subseteq S^4$ ($F_1 \cong \#_{10} \mathbb{RP}^2$, $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2} / \text{complex conjugation} = S^4$, branch set F_1).

Replace  $\times S^1 \subseteq (S^4, F_1)$ with 

$\times S^1$

π_1 of surface complements of F_n are simple, so can show they're all topologically isotopic.

To show not smoothly isotopic, show Σ_2's are not diffeomorphic

Σ_2's are obtained by replacing a $T^2 \times D^2$ with $(S^3 \setminus T_{n,n+1}) \times S^1$

Donaldson, SW, $\overset{HF}{\ldots}$ $\Rightarrow$ distinct